Statistics of transmission in one-dimensional disordered systems: Universal characteristics of states in the fluctuation tails

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We numerically study the distribution function of the conductance (transmission) in the one-dimensional tight-binding Anderson and periodic-on-average superlattice models in the region of fluctuation states where single parameter scaling is not valid. We show that the scaling properties of the distribution function depend upon the relation between the system's length L and the length l_s determined by the integral density of states. For long enough systems, $L \gg l_s$, the distribution can still be described within a new scaling approach based upon the ratio of the localization length l_{loc} and l_s. In an intermediate interval of the system's length L, $l_{\mathrm{loc}} \ll L \ll l_s$, the variance of the Lyapunov exponent does not follow the predictions of the central limit theorem and this scaling becomes invalid.

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I. INTRODUCTION

Coherent transport properties of disordered systems have been a subject of active research for the last thirty years, but complete understanding of this phenomenon even for one-dimensional models is still absent. Even though the scaling theory, put forward in the pioneering work of Ref. 1, created a successful conceptual framework for discussing the phenomenon of localization, the theoretical foundation of the scaling hypothesis itself has not yet been completely understood. One of the principal difficulties that the scaling theory of localization had to deal with from the very beginning was an absence of self-averaging of the main transport coefficients: conductance g or transmission T. Therefore, even the nature of the scaling parameter remained unclear until it was realized that the scaling hypothesis has to be applied to the entire distribution function of the conductance or transmittance.[2–5]

For one-dimensional systems Anderson *et al.*[2] suggested that the most suitable quantity for dealing with the statistical description of conductance is the Lyapunov exponent (LE), which can be defined for systems with finite length L as

$$\tilde{\gamma} = \frac{1}{L}\ln\left(1+\frac{1}{g}\right) = \frac{1}{L}\ln T. \tag{1}$$

The name ''Lyapunov exponent'' alludes to the fact that the quantity defined by Eq. (1) have the same statistical properties as the ''real'' Lyapunov exponent, i.e. the exponential growth rate $-(1/L)\ln|\psi|$ of the norm of the wave function ψ. An important property of LE is that it satisfies a multiplicative central limit theorem[6] and approaches a nonrandom limit γ when the size of the system L tends to infinity. The localization length l_{loc} of a state with energy E in the infinite system is related to γ as $l_{\mathrm{loc}} = \gamma^{-1}$. At finite L, $\tilde{\gamma}$ is a random quantity with mean value equal to γ, $\langle\tilde{\gamma}\rangle = \gamma$. The distribution of LE is the main object of research in the field of one-dimensional localization. The hypothesis of single parameter scaling (SPS) in this context means that the distribution function can be parametrized by a single parameter γ itself. As a result, it is expected that all moments of the distribution can be expressed in terms of the first moment $\langle\tilde{\gamma}\rangle$ in a universal way. For the second moment (variance) σ^2 such a relationship, as it was first conjectured by Anderson *et al.*,[2] can be presented in the form

$$\sigma^2 = \frac{\gamma}{L}. \tag{2}$$

The entire distribution function of LE for systems with finite lengths was also derived by several authors in the limit of infinitesimally weak local scattering for several models.[3,8,9] For finite L, this function was found to be non-Gaussian, but nevertheless, it depended upon a single parameter — the localization length.

Thus, in the situations when SPS holds the problem of the conductance/transmision distribution function can be considered as settled. There are spectral regions, however, where SPS fails even for locally weak disorder. These are, first of all, the regions of fluctuation states, which arise outside of the initial spectrum because of disorder. This result was first obtained numerically in Ref. 10 for a periodic-on-average system and was confirmed by an exact analytical solution of the Lloyd model (the Anderson model with the Cauchy distribution of the site energies).[11,12] Similar results were obtained numerically for the Anderson model with the box[11,12] and dichotomic[13] distributions of the site energies, and analytically for a continuous model with white noise Gaussian potential.[14] The analytical calculations of Refs. 11,12 revealed that the criterion for the validity of SPS can be presented in the form $l_{\mathrm{loc}} > l_s$, where l_s is a new scale introduced in Refs. 11,12. For the Lloyd model this scale is defined in terms of the imaginary part of the Lyapunov exponent, which, according to Thouless[15] is proportional to the integral density of states. Therefore, l_s can be presented in the form

$$l_s = \frac{a}{\sin[\pi N(E)]}, \tag{3}$$

where $N(E)$ is the integral density of states between the genuine boundary of the spectrum, the energy E is normalized by the total number of states in the system, and a is a distance between the neighboring sites. The definition of l_s in this form can be easily generalized to other models as well, and it was shown numerically that the SPS criterion based upon l_s works for such models as the Anderson model with box[11,12] and dichotomic[13] distributions of site energies, a Kronig-Penney-like model with a periodic-on-average distribution of barrier widths,[11,12] and a model of a scalar wave propagating in an one-dimensional absorbing disordered medium.[16] The case of periodic-on-average models involves a system with multiple bands, and in this case $N(E)$ must be understood as the integral density of states between a genuine boundary of the band (if the latter exist) and the energy E normalized by the total number of states in the band. More detailed discussion of this case can be found in Ref. 12. In a recent paper Ref. 17, it was shown how this criterion can be applied to the zero energy states of the Anderson model with a diagonal disorder, where the violation of SPS was observed in Ref. 18.

The criterion based on l_s replaces an original criterion put forward by Anderson *et al.*[2] that suggested that SPS exists if the stationary distribution of the phases of the reflection and transmission coefficients is uniform, and the phase relaxes to this distribution over a length, which is much smaller than the localization length. By using the hypothesis of the phase randomization (2) was rederived by many authors for a variety of different models.[7] The phase randomization was proven rigorously in some one-dimensional[5,19,20] and quasi-one-dimensional[21,22] models, but only for certain parts of the spectrum of the respective systems. At the same time, it was found that, for instance, in the Anderson model with a diagonal disorder the stationary distribution of the phase is not uniform for all values of energy E, for which $\cos^{-1}(E/2)$ is a rational fraction of π (it is assumed that in the nonrandom case all site energies in the Anderson model are set to zero, and the interaction parameter is chosen to be equal to unity). The strongest deviation of the phase distribution from the uniform one takes place in the vicinities of $E=0$ and the initial band boundaries $E=\pm 2$. While it was found that an absence of the phase randomization in both of these cases is accompanied by the violation of SPS,[10–12,18] the reference to the phase randomization as a criterion for SPS does not seem to be satisfactory. Indeed, the initial idea of the phase randomization length,[2] used to introduce the criterion for SPS, does not actually describe the way the distribution of phase becomes nonuniform. The absence of the phase randomization does not mean that the relaxation length of the distribution of phase becomes too large and exceeds the localization length. What it means is that the stationary distribution of phase, which can be reached over relatively short distances, is merely not uniform. Thus, the problem of a criterion for SPS is simply replaced by the problem of finding a criterion describing the transition between uniform and nonuniform stationary distributions of the phase. A solution for the latter problem suggested, for instance in Ref. 18, applies to only one particular model, and, actually involves different criteria for different spectral regions. In contrast, the criterion based on l_s introduced in Refs. 11,12 was proven to work for the entire spectrum of the variety of different models, and offers, therefore, a universal approach to the verification of SPS.

The violation of SPS in the spectral region of fluctuation states rises a question about the properties of the probability distribution of LE in these regions. Recently, a significant progress in this direction was achieved in Refs. 14,23. In the former paper, the first four moments of this distribution were found analytically for the Anderson model with a Gaussian white-noise potential. The authors of the latter paper used numerical simulations to develop a macroscopic scaling approach to this problem, and one which could be readily applied to a wide variety of different systems. It was shown in Ref. 23, that not only second, but also the third moment of the distribution function of LE for the Anderson tight-binding model with diagonal disorder can be fully characterized by a scaling parameter $\kappa = l_{\mathrm{loc}}/l_s$.

The objective of the present paper is to present more fully and expand the results of Ref. 23. Considering two quite different models of one-dimensional localization such as the Anderson tight-binding model with a diagonal disorder, and a model of a scalar wave propagating in a one-dimensional random superlattice, we demonstrate that one-dimensional disordered systems allows for a universal scaling description of the conductance (or transmission) distribution in the spectral regions of fluctuation states, where standard SPS does not work. In particular, we show that the scaling approach suggested in Ref. 23 describes not only the Gaussian bulk of the distribution function, but is also capable of describing the statistics of large deviations characterized by the third moment of the distribution.

The results presented in this paper are also relevant to the problem of resonant tunneling through disordered potential barriers. This problem was considered in the pioneering work by Lifshits and Kirpichenkov[24] for quantum particles incident on a three-dimensional barrier, and was later studied in many subsequent papers (see reviews in Ref. 25,26). Mostly, these works were concerned with tunneling through three-dimensional barriers with the dimension in the propagating direction much smaller than in the perpendicular directions. Even though the resonant tunneling is in many aspects a quasi-one-dimensional process,[24] the transport in the pure one-dimensional models significantly differs from the situation described above. First of all, in one-dimensional case all states of an infinite sample are localized. Respectively, transmission through a finite, but longer than the typical localization length, system can be described as a resonant under-barrier tunneling at any energy regardless to its position in the initial spectrum. Therefore, the difference between transport in the region of states from the initial allowed bands and the fluctuation states, is not as clear as in three-dimensional situations. Therefore, the problem of transport via fluctuation states was not considered as a separate problem in the area of one-dimensional localization until very recently.[11,12] Second, the main quantity of interest in the case of three dimensional barriers is the total transmittance across

the entire area of the barrier, which is determined by the sum of individual transmissions through independent quasi-one-dimensional channels or filaments.[24,26] This quantity approaches a nonrandom limit when the area of the barrier tends to infinity. In a pure one-dimensional case the self-averaging quantity is the Lyapunov exponent, which becomes nonrandom when the length of the system becomes infinite. In a sense, the pure one-dimensional case is an opposite limit to the one considered for three-dimensional barriers. At the same time, solutions of the one-dimensional problem can be used to describe barriers whose lengths are larger than the typical localization length of individual channels.

Another important application of the problem studied in this paper lies in the field of random lasing, which has become an area of active research.[27] It is anticipated that using localized modes of a strongly scattering disordered medium, one can obtain very low-threshold lasing. Disordered photonic crystals, which support fluctuation photon states in the band gaps of the underlying periodic structures, can play an important role in achieving this objective.[28,29] The results presented in this paper will help to understand the unusual statistical properties of the lasing threshold and the nature of lasing modes in such structures.

II. MODELS AND TECHNICAL DETAILS

In this paper we study two models of one-dimensional Anderson localization: a classical Anderson tight-binding model with a diagonal disorder, and a scalar wave propagating in a one-dimensional random superlattice. The Anderson model is described by the equation of motion

$$\psi_{m+1}+\psi_{m-1}+(U_m-E)\psi_m=0, \tag{4}$$

where random on-site energies U_m are described by a uniform probability distribution

$$P(U_m)=\begin{cases}\dfrac{1}{2U}, & |U_m|<U,\\ 0, & |U_m|>U.\end{cases}$$

The propagation of a scalar wave is described by a regular wave equation

$$\frac{d^2\psi}{dx^2}+k^2\epsilon(x)\psi=0, \tag{5}$$

with a piece-wise dielectric function, corresponding to a superlattice consisting of two types of layers with dielectric constants ϵ_1 and ϵ_2, respectively. The width of the layers of the first kind is kept constant and is equal to d_1, while the width of the layers of the second type was chosen from a random distribution. In this paper, we report the results for (i) d_2 uniformly distributed in the interval $\langle d_2\rangle-\delta,\langle d_2\rangle+\delta$ (uniform distribution) and (ii) d_2 taking one of two equally probable values $-\delta/\sqrt{3}$ and $+\delta/\sqrt{3}$ (dichotomic distribution).

Both these models can be studied using the transfer matrix approach, in which the propagation of the excitation along the system is presented in the following form:

$$v_{m+1}=T_m v_m, \tag{6}$$

where v_m is a two-dimensional state vector, which presents the state of the system at the mth site (or mth interface between the layers) and T_m is the transfer matrix describing the change of this state at one discreet step. For the Anderson model the state vector and the transfer matrix have the following forms, respectively:

$$v_m=\begin{pmatrix}\psi_m\\ \psi_{n+1}\end{pmatrix}, \tag{7}$$

$$T_m=\begin{pmatrix}E-U_m & -1\\ 1 & 0\end{pmatrix}. \tag{8}$$

For the second model the state vector can be defined as

$$v_m=\begin{pmatrix}\psi_m\\ \psi_m'\end{pmatrix}, \tag{9}$$

where ψ_m and ψ_m' are the values of the wave function $\psi(x)$ and its derivative at the mth interface between the layers. The transfer matrix in this case takes the form

$$T_m=\begin{pmatrix}\cos(k_m d_m) & (1/k_m)\sin(k_m d_m)\\ -k_m\sin(k_m d_m) & \cos(k_m d_m)\end{pmatrix}, \tag{10}$$

where $k_m=k\sqrt{\epsilon_m}$. The most important property of the transfer matrices is that the transfer matrix $\mathbf{T_M}$ describing the evolution of the initial state vector across the M sites (slabs) is equal to the product of the one-step matrices

$$\mathbf{T_M}=\prod_1^M T_m. \tag{11}$$

Using the transfer matrices, we calculate the finite size LE, which for both models is defined as

$$\tilde{\gamma}=\frac{1}{L}\ln\frac{\|\mathbf{T_M}v_0\|}{\|v_0\|}, \tag{12}$$

where L characterizes the total length of the system. For the Anderson model, $L=M$ if the distance between adjacent sites is chosen as a unit of length, and for the wave equation, L is a sum of the lengths of all slabs, and is a random quantity.

We calculate LE iteratively using Eq. (12) starting with an arbitrary initial vector v_0. The resultant vector is renormalized after every ten iterations in order to avoid any loss of accuracy.[30] Since we are interested in statistics of finite size LE, we do not try to find its limiting value for $L\to\infty$. Instead, we keep the size of the system fixed while calculating $\tilde{\gamma}$ for different realizations of our systems. At the same time, since we are interested in asymptotic properties of the distribution, we consider only sufficiently long systems, for which

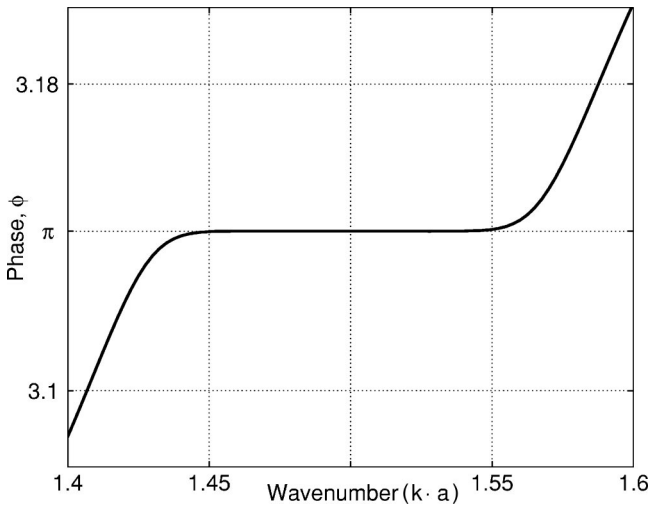

FIG. 1. The dependence of the phase near the band gap region ($1.44<ka<1.56$) separating the first and the second bands in the superlattice model. d_2 was taken from a uniform distribution with $\delta=0.1$, $L/a\simeq 10^6$.

$L \gg l_{\mathrm{loc}}$, where the localization length l_{loc} is defined through the average value of LE as $l_{\mathrm{loc}}=\langle \tilde{\gamma} \rangle^{-1}$.

Another quantity of interest in this work is the length l_s, which is expressed in terms of the integral density of states $N(E)$, Eq. (3). For the Anderson model $N(E)$ can be computed with the help of the node-counting theorem.[31] Starting with an arbitrary initial vector and the energy values $E<-2-U$, which are certainly outside of the energy spectrum of the system, we counted how many times the sign of the wave function changes over the length of the system for different values of E. Each new node corresponds to a new state of the system.[31]

For the random superlattice model we find it more convenient to use the phase formalism described, for instance, in Ref. 5. Within this formalism the density of state is expressed in terms of the phase variable $\phi=\tan^{-1}(\psi'/\psi)$. In the case of systems with a single band spectrum, this phase changes between 0 and π when E sweeps the spectrum of the system from one band boundary to the other. In the superlattice, the spectrum of the wave in the absence of disorder consists of multiple bands. In this case, the phase increases by π across every allowed band, and stays constant and equal to $n\pi$, inside any nth forbidden band. If disorder in our model is not too strong, the regions of the constant phase are preserved even in the presence of random fluctuations Fig. 1, and can be used for identifying the fluctuation boundaries of the bands in the disordered system. Then we can introduce a density of states $N(E)$ for a *single band*, which is normalized to change from 0 to 1, when energy spans from one fluctuation boundary to another. $N(E)$ normalized this way is substituted in Eq. (3) in order to calculate l_s for the superlattice model. When disorder becomes stronger the regions of constant phase disappear, and the notion of the single band density of states becomes meaningless. In our calculations we always make sure to avoid such situations.

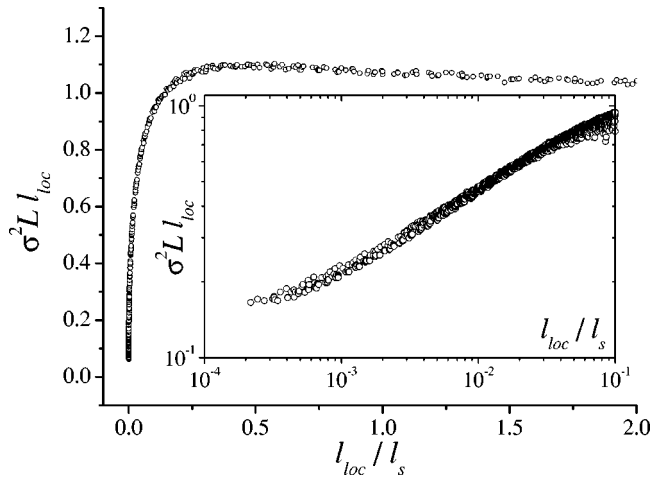

FIG. 2. Typical dependence of the scaling parameter τ on κ for the Anderson model. The width of the distribution of disorder changes from $U=0.08$ to $U=0.16$. Curves corresponding to different values of the width are not distinguishable. In the inset the region of small κ is shown in the log-log scale.

III. SCALING DESCRIPTION OF THE MOMENTS OF THE DISTRIBUTION FUNCTION

It was shown in Refs. 11,12 that the variance σ^2 of the Lyapunov exponent in the Lloyd model can be conveniently described in terms of a relationship between two scaling variables τ defined as

$$\tau=\frac{\sigma^2 L}{\gamma} \tag{13}$$

and κ, defined as

$$\kappa=\frac{l_{\mathrm{loc}}}{l_s}. \tag{14}$$

In this paper we show that the variance of LE in more generic models can also be described in terms of the scaling function $\tau(\kappa)$.

In order to demonstrate this result we computed σ^2 and l_s for different values of the energy, strength of disorder, and length of the system for both models under considerations. The results of these calculations were presented in the form of the function $\tau(\kappa)$, which is shown in Figs. 2 and 3 for the Anderson model and the superlattice model, respectively. The data included in these figures correspond to systems with $L \gg l_s, l_{\mathrm{loc}}$. The first important result revealed by this figures is that all the data lie on a single curve, when expressed in terms of the variables τ and κ for both models. This result confirms our general conjecture that the second moment of the distribution function of LE can be universally described in terms of variables τ and κ regardless the microscopical nature of the models under consideration. While the form of the function $\tau(\kappa)$, may differ for different models, its essential qualitative properties show a degree of universality: $\tau(\kappa)=1$ for $\kappa>1$, and it steeply decreases for $\kappa \ll 1$. We are most interested here in the latter region, where the fluctuation states arise. For the Lloyd model $\tau=(\pi/2)\kappa$

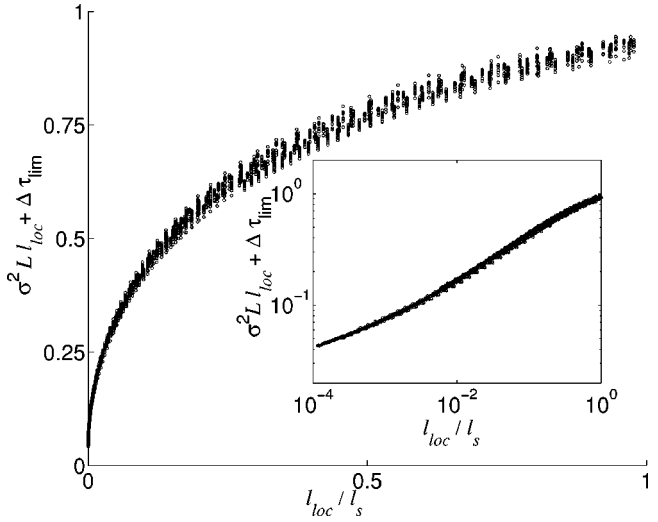

FIG. 3. Dependence of the scaling parameter τ on κ for the dichotomic distribution (superlattice model) of d_2 with $\delta=0.1$, 0.125, 0.15, 0.175, and 0.2. For every value of disorder we took 17 lengths L, ranging from 320 to 20 000 layers. Different offset value of $\tau_{\rm lim}$ was compensated (see text). In the inset the same is shown in the log-log scale.

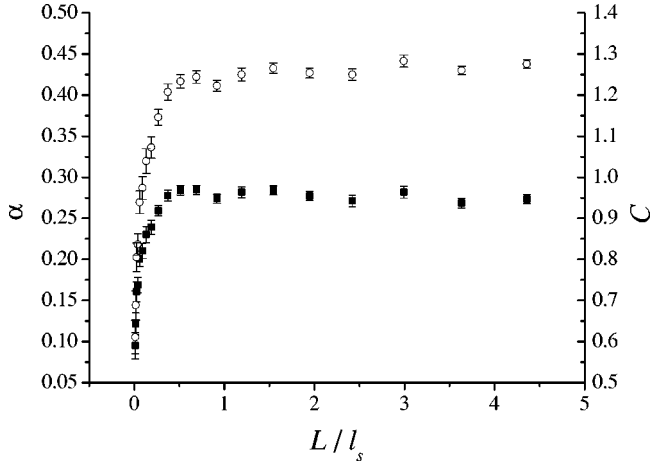

FIG. 4. Dependence of the index of the scaling parameter α (filled squares, left axis) and the factor C (circles, right axis) on L/l_s for the Anderson model.

for $\kappa \ll 1$,[11,12] while in the models studied in this paper the dependence of τ upon κ is much steeper. In order to obtain a better insight into the properties of $\tau(\kappa)$ for small κ, we replotted our numerical data in log-log coordinates (see insets in Figs. 2 and 3). Before interpreting these figures we have to note that unlike the case of the Lloyd model, where $\tau(0)=0$, in the models considered here $\tau(0)$, while very small, is not equal to zero. The reason for this is the small fluctuations of the LE due to nonresonance tunneling through a random barrier, which contributes to τ at the fluctuation spectrum boundary where $\kappa=0$. This small contribution is model specific, and in the Anderson model it can be neglected everywhere with exception of a small neighborhood of the fluctuation spectrum boundaries. This can be seen from the fact that while κ changes over at least two orders of magnitude, the data for the Anderson model (inset in Fig. 2) form a straight line with exceptions of a few points corresponding to very small values of κ. According to these results, $\tau(\kappa)$ has the form

$$\tau = C\kappa^{\alpha} + \tau_{\rm lim}, \tag{15}$$

where $\tau_{\rm lim}$ stands for the nonuniversal correction discussed above. In the superlattice model the value of $\tau_{\rm lim}$ is more significant, and therefore has to be compensated. In order to estimate coefficients C and κ, we select only those data for which $l_s < L$ and use linear regression. The results of the fit are presented in Table I. These results demonstrate that while the nature of the scaling parameters is universal for both models, the numerical values of the respective parameters are model dependent. An interesting question is whether the values of C and α depend upon the type of statistics of the respective random parameters of our models (site energy for Anderson model, the layer width for the superlattice model). In the case of a superlattice model we found that the change in statistics (from the box to dichotomic distribution) did not affect the values of the coefficients C and α. For the Anderson model with the dichotomic distribution of the site energies the results were inconclusive. Strong noise in the data for the dichotomic process prevented us from positively establishing equivalency of the coefficients for the two different types of statistics.

TABLE I. Parameters C and α from Eq. (15) for different models.

	Anderson model	Superlattice models
C	1.27	1.08
α	0.27	0.40

In the region of fluctuation states, a new intermediate regime of lengths L, in which $l_{\rm loc} \ll L \ll l_s$ appears. This regime does not exist for in-band states. It is natural to anticipate that the scaling behavior of our systems in this regime would change. In order to study this question, we divided our data in groups according to the value of L/l_s, including points with $L/l_s > 1$ as well as with $L/l_s < 1$. Carrying out statistical analysis of the data for fixed values of L/l_s we were able to obtain dependencies of the parameters C and α on L/l_s; the respective results are presented in Figs. 4 and 5. First of all, we would like to note that these dependencies saturate to values presented in the Table I for $L/l_s > 1$. This confirms our assumption that in this regime τ depends upon a single parameter κ.

For shorter systems, however, a new parameter, L/l_s emerges. For the Anderson model we were able to show that $\alpha(L/l_s)$ is best described by the logarithm $\alpha(L/l_s) \sim \ln(l_s/L)$, which means that the variance of the Lyapunov exponent σ^2 in this regime demonstrates an anomalous scaling with the length of the system L:

$$\sigma^2 \propto \frac{1}{L l_{\rm loc}} \exp[\alpha(L/l_s)\ln\kappa] \propto L^{-(1+\ln\kappa)}. \tag{16}$$

It is interesting to note that when κ decreases, $1+\ln\kappa$ may become negative resulting in σ^2 increasing with L. This be-

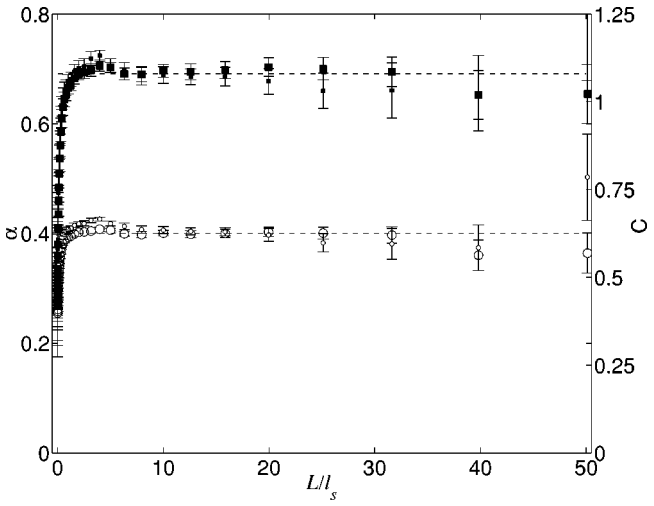

FIG. 5. The exponent of the scaling parameter α (filled squares, left axis) and the factor C (circles, right axis) as functions of L/l_s for the superlattice model. Large and small symbols correspond to dichotomic and box distribution of d_2, respectively.

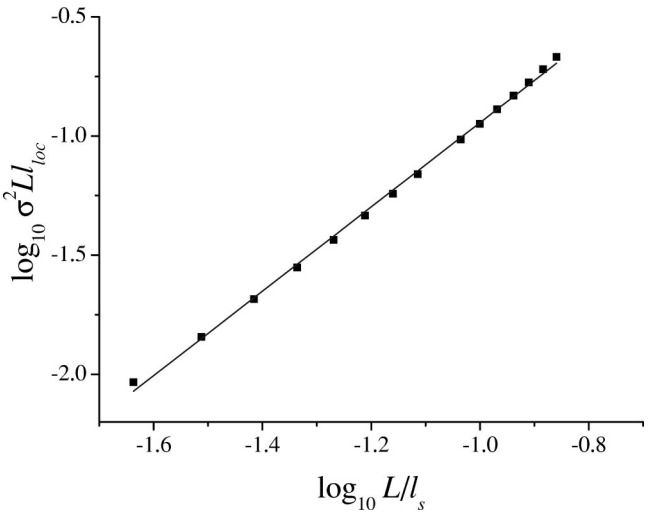

FIG. 6. The logarithm of the scaling parameter τ for the Anderson model as a function of $\log_{10}L/l_s$ for intermediate values of energy when l_s is not too large. Points are the result of numerical calculations and the straight line is a linear fit.

havior can be qualitatively understood from the following arguments: The condition $L \ll l_s$ means that for the most of the realizations of the random potential no states exist in the energy interval under discussion. The transmission through such realizations fluctuates rather weakly. The greatest contribution to the transmission fluctuations is given by those few realizations that can support at least a single state. The probability for such realizations to arise grows when the length of the system increases, resulting in the respective increase of σ^2. This behavior, of course, breaks down for very large values of l_s, which correspond to states close to the genuine spectral boundary, because for these states σ^2 is determined by a nonuniversal correction to τ given by $\tau_{\lim}$.

The behavior of σ^2 given by Eq. (16) can be confirmed by plotting directly the function $\sigma^2(L)$ for energies from the band gap. Figure 6 presents such a plot for the Anderson model for the value of κ equal to $\kappa=0.2$. It demonstrates a good agreement with Eq. (16): the slope of the curve was found to be equal to 1.77, while an estimate for this slope from Eq. (16) gives 1.78. It should be noted, however, that the regime described by Eq. (16) exists in a relatively narrow interval of energies, at least for the Anderson model with the box distribution. The reason for this is that l_s grows very fast in the region of fluctuation states when the energy is shifted toward the fluctuation spectrum boundary. Very large l_s means that only few realizations of our system support at least a single state. Therefore, for most realizations transmission occurs via nonresonant under-barrier tunneling. The statistics of the transmission for this subset of realizations is determined by the localization length alone (l_s is exact infinity for these realizations). As a result, we have a competition between a small number of realizations, supporting states, for which fluctuations of the Lyapunov exponent are large and grow with the length, and the majority of realizations, in which σ^2 is small, and decreasing with length. At very large l_s the contribution to σ^2 from the representative realizations becomes larger than the contribution from the resonant realizations, and Eq. (16) fails. In this case, an asymptotic behavior of σ^2 is again controlled by the localization length alone, as it can be seen in Fig. 7, where $\sigma^2 L$ saturates at L much smaller than l_s.

The assumption about the Gaussian form of the distribution of LE is the result of the central limit theorem, and strictly speaking is true only asymptotically when $L \to \infty$. At finite L the distribution function deviates from the Gaussian form even in the regime when SPS holds.[3,8,9] However, it was found in Refs. 14,23 that this deviation, as measured by the magnitude of the third and higher moments, increases significantly in the vicinity of the band boundary of the initial spectrum. This result was obtained analytically for the white-noise potential in Ref. 14. The first study of the scaling properties of the third moment was reported in Ref. 23. In

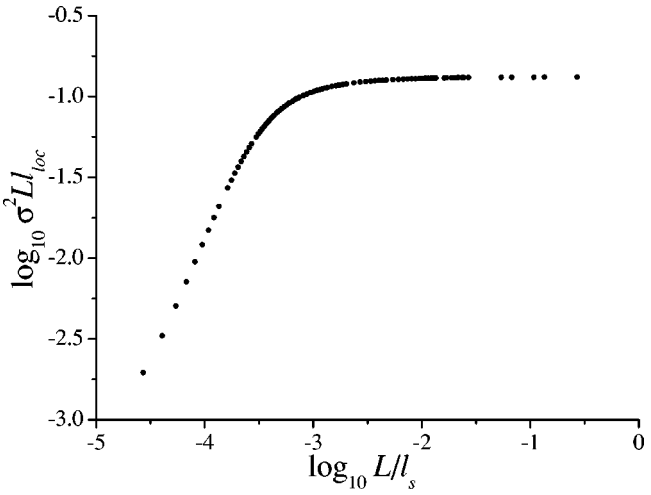

FIG. 7. The logarithm of the scaling parameter τ for the Anderson model as a function of $\log_{10}L/l_s$ for energies corresponding to extremely large values of l_s. The saturation occurs at the length close to the localization length.

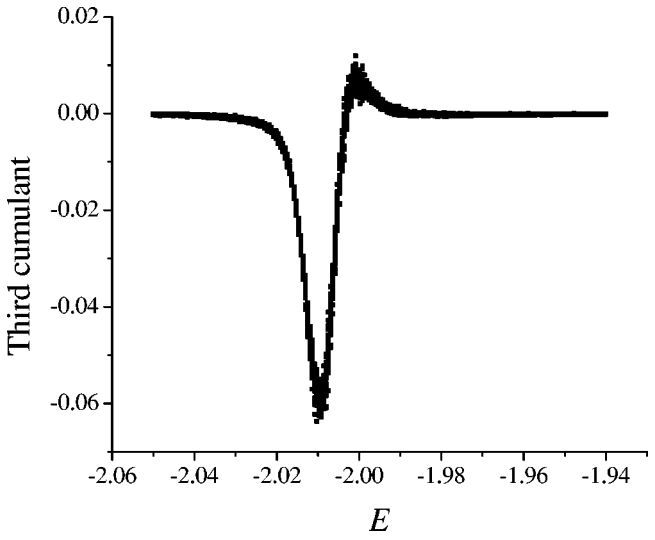

FIG. 8. Dependence of the renormalized third cumulant ϱL^2 on energy in the vicinity of the band edge of a pure system ($U=0.05$) for the Anderson model.

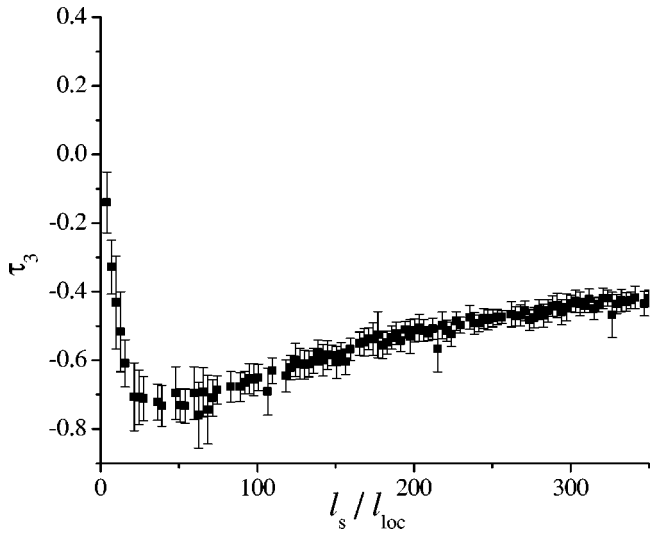

FIG. 9. Dependence of the parameter $\tau_3=\varrho L^2 l_{\rm loc}$ on $\kappa^{-1}=l_s/l_{\rm loc}$ (Anderson model) for a set of different widths of the distribution of the potential $0.001<U<0.21$. Error bars show the dispersion of the results of numerical simulations near a mean values shown by squares.

this part of the paper we expand scaling analysis of Ref. 23 to the superlattice model, and compare the results obtained for these two models. We consider the scaling properties of the third cumulant $\varrho=\langle(\gamma-\langle\gamma\rangle)^3\rangle$, which characterizes the asymmetry or skewness of the distribution function. Figure 8 shows the energy dependence of the third moment for the Anderson model. It is seen that this moment significantly grows in the vicinity of the initial band boundaries of both models, which means that the significant deviation of the distribution function of LE from the Gaussian form in the region, where traditional SPS violates is a universal phenomenon.

To analyze scaling properties of the third cumulant we consider the dimensionless parameter

$$\tau_3=\varrho L^2 l_{\rm loc}. \tag{17}$$

The dependence of τ_3 on κ for the Anderson model and superlattice is shown in Figs. 9 and 10, respectively.

One can see from these figures that while data for the parameter τ_3 are rather noisy, it shows a relatively good scaling behavior as a function of the single parameter κ for both models. This fact itself is quite remarkable since it demonstrates that even in the region, where the distribution function of LE deviates significantly from the Gaussian form, it can still be characterized by two parameters within the scaling procedure suggested here.

The better data quality for the superlattice model allowed for a more thorough study of the third moment. The insert in Fig. 10 shows a good scaling behavior similar to Eq. (15):

$$-\tau_3=C_3\kappa^{\alpha_3}+\tau_{3,\rm lim}. \tag{18}$$

The limiting value $\tau_{3,\rm lim}$ was substantially smaller than $\tau_{\rm lim}$, so no explicit correction was needed to obtain Fig. 10.

For intermediate lengths, $l_{\rm loc}\ll L\ll l_s$, we analyzed data using approach similar to that employed to obtain Figs. 4 and 5. For fixed values of L/l_s we obtained dependencies of the parameters C_3 and α_2 on L/l_s (Fig. 11), and found the saturated values of $C_3=0.73$ and $\alpha_3=0.52$ – the same for both dichotomic and box distributions.

IV. COMPARISON WITH THE GAUSSIAN WHITE NOISE MODEL

It is well known that under certain circumstances statistical properties of one-dimensional disordered systems in the vicinity of the band edges of the initial spectrum can be universally described by replacing an actual random potential by a Gaussian white noise potential.[5] One of the manifestations of this fact is that the statistical properties of LE in

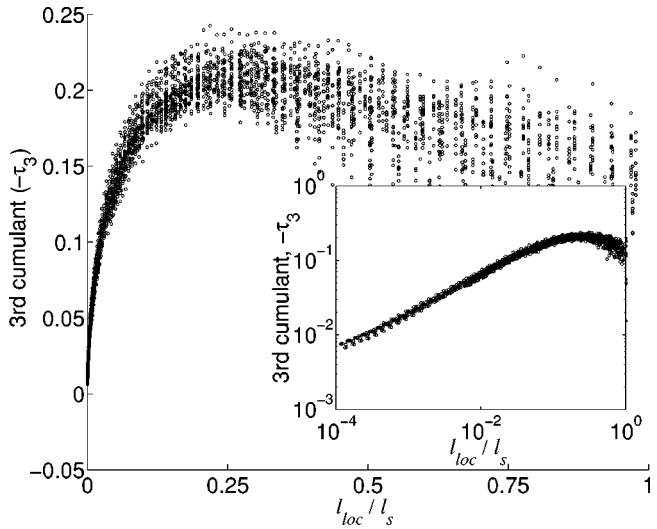

FIG. 10. Dependence of τ_3 on $l_{\rm loc}/l_s$ for dichotomic distribution (superlattice model) of d_2 with $\delta=0.1$, 0.125, 0.15, 0.175, and 0.2. For every value of disorder we took 17 length L, ranging from 320 to 20 000 layers. On the insert the same is shown in log-log scale.

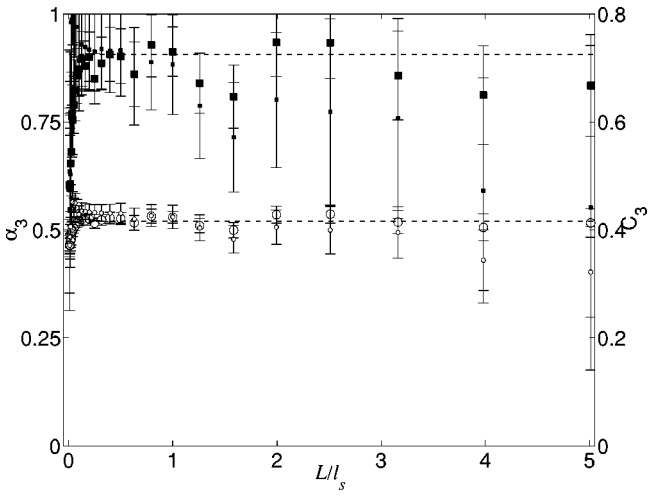

FIG. 11. α_3 (filled squares, left axis), the exponent of τ_3, and factor C_2 (circles, right axis) as functions of L/l_s for superlattice model. Large and small symbols correspond to dichotomic and box distribution of d_2, respectively.

the Anderson model with the box distribution of the site energies[14,19,20,32] are very similar to those of the continuous model with the Gaussian white-noise potential,[5] and are characterized by the same scaling parameter $E/D^{2/3}$, where E is the energy counted from the initial band boundary and D is the variance of the random potential. It was noted in Ref. 12 that the scaling parameter κ is a single-valued function of the Gaussian scaling parameter $E/D^{2/3}$ for the white-noise model, so that in this case these two parameters are equivalent to each other. An important question now arises: whether the apparent universality of the scaling description, suggested in this paper, is a mere consequence of the fact that in the region of the fluctuation states all models can be reduced to the Gaussian model, or this universality reveals more fundamental properties of this spectral region. This question was partially discussed in the Ref. 17, in which it was shown that the behavior of the second moment of the LE in the vicinity of $E=0$ of the Anderson model obeys the scaling description in terms of the parameter κ, while the Gaussian approximation certainly does not work in this part of the spectrum. In this paper, we address this question considering regions of the fluctuation states in the superlattice model.[33] The inset to Fig. 12 shows the plot of the parameter τ versus the Gaussian scaling parameter $(k-k_i)/D^{2/3}$ $(D\propto\delta^2)$, where k_i is the dimensionless frequency of one of the initial band boundaries of the superlattice for several values of the disorder. Moreover, we included the frequencies from the upper edge of the first band and the lower edge of the second band. We found that instead of $E/D^{2/3}$ predicted by the Gaussian white noise model, our data are better scaled with the parameter $E/D^{1/2}$. One can see from Fig. 12, that while the Gaussian scaling fails, the function $\tau(\kappa)$ discussed in the previous section of the paper gives the best scaling description of this model as well as of the Anderson model. We can conclude, therefore, that the scaling parameter κ retains its universal significance beyond the validity of the white-noise approximation.

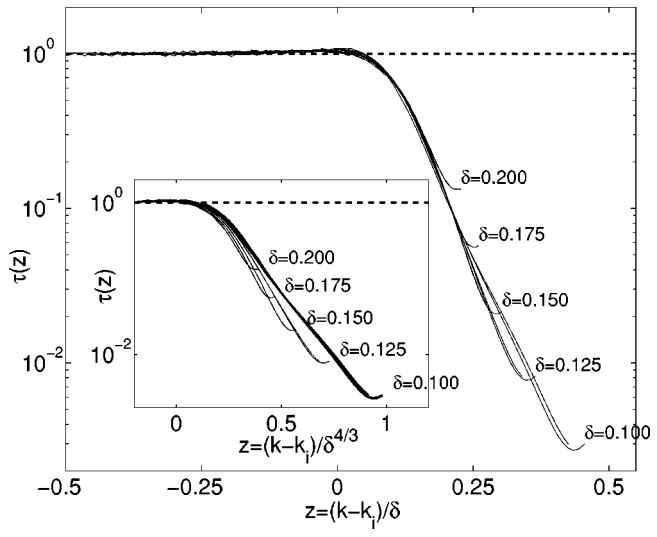

FIG. 12. Normalized variance of LE τ plotted versus parameter $(k-k_i)/\delta$ demonstrates a good scaling. Scaling with $(k-k_i)/\delta^{4/3}$, predicted by the Gaussian white noise model, shown in the inset, fails. The data was generated in the superlattice model with the box distribution for five values of $\delta=0.1$, 0.125, 0.15, 0.175, and 0.2, $L=20\,000$. We included the frequencies from two band edges — the upper edge of the first band and the lower edge of the second band. Altogether, the band gap region between the first and the second bands is covered entirely.

V. CONCLUSION

In this paper we studied scaling properties of the distribution function of the Lyapunov exponent for two one-dimensional disordered models: the Anderson model with diagonal disorder, and the model of a scalar wave propagating in a random superlattice. The main result of the paper is that in the region of band-edge and fluctuation states, where simple SPS fails, the distribution function can be described by two independent scaling parameters: the localization length l_{loc} and an additional length l_s, introduced in Refs. 11,12, which is related to the integral density of states. It is interesting that not only the second moment of the distribution is described by these two parameters, but so also is the third moment. This means that even though in the region of fluctuation states the form of the distribution function strongly deviates from the Gaussian, it still can be described within the suggested two-parameter scaling approach.

Among the other results of the paper we would like to note the detailed study of the properties of the variance and the third moment of LE in the region of fluctuation states. We showed that both, the normalized variance and the third cumulant presented by the scaling functions τ and τ_3, demonstrate a power law dependence upon the scaling parameter κ. Parameters of this power law dependence were found to depend weakly upon the type of statistics used to characterize our random systems, but are different for the Anderson model and the superlattice model. When the length of the system becomes smaller than l_s, we showed that the scaling behavior of σ^2 deviates significantly from the central limit theorem behavior even when L remains much bigger than the localization length.

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