SUPPLEMENTAL MATERIAL
Position-dependent diffusion of light in disordered waveguides

Alexey G. Yamilov[1*], Raktim Sarma[2], Brandon Redding[2], Ben Payne[1], Heeso Noh[2,3] & Hui Cao[2,4†]
[1]*Department of Physics, Missouri University of Science & Technology, Rolla, Missouri 65409, USA*
[2]*Department of Applied Physics, Yale University, New Haven, Connecticut 06520, USA*
[3]*Department of Nano and Electronic Physics, Kookmin University, Seoul 136-702, Korea*
[4]*Department of Physics, Yale University, New Haven, Connecticut 06520, USA*

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MAPPING FROM 3D TO 2D WAVE EQUATION

The random waveguide, c.f. Fig. [1] in the main text, is made of perforated silicon membrane, sandwiched between air and silica. The dielectric constant of this 3D system is $\epsilon(x,y,z) = [n(x,y,z)]^2$, and the wave equation is

$$\left\{\nabla^2_{3D} + [kn(x,y,z)]^2\right\} E(x,y,z) = 0. \quad (1)$$

The transformation of the above wave equation to 2D wave equation involves two assumptions – effective index and effective absorption approximations.

Because the refractive index $n(x,y,z)$ of our system is not factorizable, the above wave equation does not separate exactly into a normal (x-axis) and in-plane (y,z-axes) equations. The transformation from the 3D wave equation to a 2D wave equation for y,z coordinates is known as effective index approximation. This involves replacing $n(x,y,z)$ by an effective index $\tilde{n}$ that depends only on y,z. The 2D wave equation is

$$\left\{\nabla^2_{2D} + [k\tilde{n}(y,z)]^2\right\} \tilde{E}(y,z) = 0. \quad (2)$$

The value of $\tilde{n}(y,z)$ is chosen to be one within the air holes and n_d in the dielectric. The value of n_d can be found from a procedure described e.g. in Ref. [1]. The most important limitations of this approach are:
(i) $\tilde{n}$ varies with frequency even if $n(x,y,z)$ is independent of frequency;
(ii) $\tilde{n}$ is a real number and it does not account for the out-of-plane leakage of light from the membrane.

(i) is not an issue in our experiments using continuous-wave monochromatic light.

(ii) can be mitigated with another approximation: the out-of-plane scattering loss can be accounted for by adding an imaginary part to the effective dielectric constant, $\tilde{\epsilon}(y,z) = (1+i\alpha)[\tilde{n}(y,z)]^2$, where α is the effective absorption coefficient. For a periodically perforated membrane, the effective absorption is not always justified because the out-of-plane loss, unlike the absorption, can be a non-local process. This is because long-range correlation of light fields in a periodic array of scatterers makes the waves scattered from different locations phase coherent and they interfere in the far field. However, in a random array of scatterers, the fields are correlated only within a distance of the order one transport mean free path ℓ [2, 3], and waves from different coherent regions $\ell \times \ell$ have uncorrelated phases. Since there are a large number of such coherence regions $\ell \times \ell$ in our waveguides $W \times L$, the overall leakage may be considered incoherent and treated effectively as absorption.

CALCULATION OF POSITION-DEPENDENT DIFFUSION COEFFICIENT $D(z)$

In the ab-initio numerical simulation, we consider a monochromatic scalar wave $E(\mathbf{r})e^{-i\omega t}$ propagating in a 2D volume-disordered waveguide of width W and length $L \gg W$. The wave field $E(\mathbf{r})$ obeys the 2D Helmholtz equation:

$$\left\{\nabla^2 + k^2\left[1+\delta\epsilon(\mathbf{r})\right]\right\} E(\mathbf{r}) = 0. \quad (3)$$

Here $k = \omega/c$ is the wavenumber and $\delta\epsilon(\mathbf{r}) = (1 + i\alpha)\delta\epsilon_r(\mathbf{r})$, where $\delta\epsilon_r(\mathbf{r})$ describes the random fluctuation of the dielectric constant, and $\alpha > 0$ denotes the strength of dissipation. The system is excited from one open end ($z = 0$) of the waveguide (extending from $z = 0$ to $z = L$) by illuminating each of the guided modes with a unit flux. The wave field $E(\mathbf{r})$ throughout the random medium is computed with the transfer matrix method for a given realization of disorder[4]. From $E(\mathbf{r})$ we calculate the energy density $\mathcal{W}(z)$ and the flux $J_z(z)$ along the z axis (parallel to the waveguide axis). These two quantities are averaged over the cross section of the waveguide at each z and give the diffusion coefficient:

$$D(z) = -\langle J_z(z)\rangle / \left[d\langle\mathcal{W}(z)\rangle/dz\right], \quad (4)$$

where the averages $\langle\ldots\rangle$ are taken over a statistical ensemble of 10^6 disorder realizations.

In order to compare our numerical results for $D(z)$ with the self-consistent theory of localization, we need to have

*e-mail:yamilov@mst.edu
†e-mail:hui.cao@yale.edu

the value of the diffusion coefficient without renormalization due to the wave interference effects $D_0 = v\ell/2$. To estimate the transport mean free path ℓ in our model we perform a set of simulations for different waveguide lengths L, exploring both the regime of diffusion $L < \xi$ and that of Anderson localization $L > \xi$. We computed numerically the conductance g as the sum of transmission coefficients from all incoming to all outgoing waveguide modes. The dependencies of the average $\langle g \rangle$ and variance $\mathrm{var}(g)$ on L are fitted by the analytical expressions obtained by Mirlin in Ref. [5] using the supersymmetry approach with ℓ being the only fit parameter. To find the diffusive speed v, we use the definition of diffusive flux in the forward $(+z)$ direction $J_z^{(+)}(z)$ and the backward $(-z)$ direction $J_z^{(-)}(z)$ with respect to the propagation direction[6]

$$\langle J_z^{(\pm)}(z)\rangle = (v/\pi)\langle \mathcal{W}(z)\rangle \mp (D(z)/2)d\langle \mathcal{W}(z)\rangle/dz. \quad (5)$$

Combining the two components, we find the diffusive speed

$$v = 2\left(\langle J_z^{(+)}(z)\rangle + \langle J_z^{(-)}(z)\rangle\right)/\langle \mathcal{W}(z)\rangle. \quad (6)$$

Dashed lines in Fig. 1 depict $D(z)$ found in equation (4) normalized by D_0.

In the dissipative random waveguides, the characteristic dissipation time τ_a is determined numerically using the condition of flux continuity $d\langle J_z(z)\rangle/dz = (1/\tau_a)\langle \mathcal{W}(z)\rangle$. The desired diffusive dissipation length $\xi_{a0} = \sqrt{D_0\tau_a}$ can be obtained by the proper choice of α in equation (3).

SELF-CONSISTENT THEORY OF LOCALIZATION

The self-consistent theory starts with the Green's function $G(\mathbf{r},\mathbf{r}')$ of equation (3) with $\delta\epsilon(\mathbf{r}) = \delta\epsilon_r(\mathbf{r}) + i\alpha$. In a random waveguide, the disorder-averaged function $\hat{C}(\mathbf{r},\mathbf{r}') = (4\pi W D_0/cL)\langle|G(\mathbf{r},\mathbf{r}')|^2\rangle$ obeys self-consistent equations in a dimensionless form[4, 7]:

$$\left[\left(\frac{L}{\xi_{a0}}\right)^2 - \frac{\partial}{\partial\zeta}d(\zeta)\frac{\partial}{\partial\zeta}\right]\hat{C}(\zeta,\zeta') = \delta(\zeta-\zeta'), \quad (7)$$

$$\frac{1}{d(\zeta)} = 1 + \frac{2L}{\xi}\hat{C}(\zeta,\zeta), \quad (8)$$

where $d(\zeta) = D(\zeta)/D_0$ and all position-dependent quantities are functions of the longitudinal coordinate $\zeta = z/L$. The quantity $\hat{C}(\zeta,\zeta)$, which renormalizes the diffusion coefficient, is proportional to the return probability at ζ. Assuming first that $d(\zeta) \equiv 1$, equations (7,8) are solved by iteration with the boundary conditions:

$$\hat{C}(\zeta,\zeta') \mp \frac{z_0}{L}d(\zeta)\frac{\partial}{\partial\zeta}\hat{C}(\zeta,\zeta') = 0 \quad (9)$$

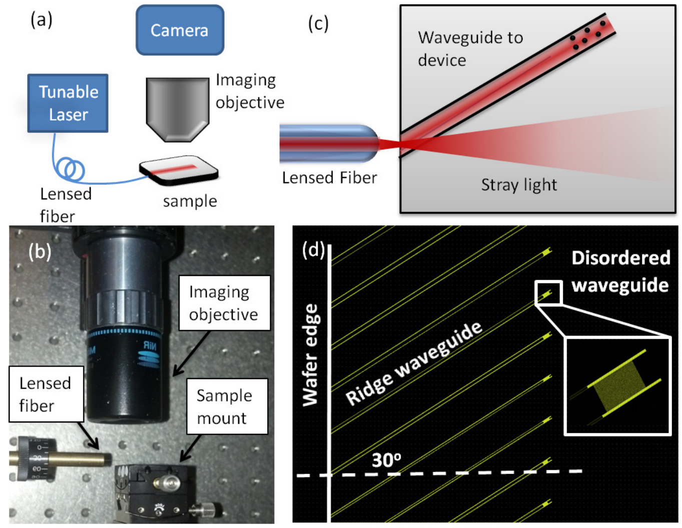

Figure S 1: Optical measurement setup: (a) Schematic of experimental setup for measuring light transport inside the random waveguide. (b) Photograph of the experimental setup. (c) Schematic of the sample layout showing the ridge waveguides coupling the probe light from the edge of the wafer to the random waveguides with photonic crystal sidewalls. (d) Layout of the fabricated structures studied experimentally.

at $\zeta = 0$ and $\zeta = 1$. The $z_0 = (\pi/4)\ell$ is the so-called extrapolation length[6].

After the self-consistent solution of equations (7-9) has been found, we find the intensity distribution inside the sample by replacing the delta-function source in equation (7) with $(L/\ell)\exp[-\zeta/(\ell/L)]$. This source term represents the exponential attenuation of the incident ballistic signal.

EXPERIMENTAL DETAILS

Optical measurement setup: The experimental setup for optical characterization is shown in Fig. S1(a). We used a single-mode polarization-maintaining fiber to deliver the probe light into a silicon ridge waveguide on a SOI substrate. The fiber was tapered at the end to focus the laser beam to a spot of diameter ~ 2.5 μm at the edge of the wafer. The ridge waveguide had the same width as the random waveguide it was connected to, which varied from 5 micron to 60 micron [Fig. S1(b)]. However, the height of the silicon waveguide was merely 220 nm, so some of the input light did not couple into the waveguide; instead it propagated above or below the waveguide. To avoid such stray light, the ridge waveguide was tilted by 30 degrees with respect to the incident direction of the light from the fiber (approximately normal to the edge of the wafer). The ridge waveguide was made 2.5 mm long, so that the random waveguide structure is far from the direct path of the stray light. In addition, uniform illumination of the front surface of the random structure inside the waveguide was ensured by positioning the tapered fiber approximately at the center of the input facet of the ridge waveguide. The spatial distribution of light intensity over the sample was imaged by an objective lens onto an IR CCD camera [not shown in Fig. S1(a)].

Design of photonic crystal walls for 2D waveguides: The triangular lattice of air holes that form the sidewalls of the random waveguide were designed to have a 2D photonic bandgap for TE polarized light in the wavelength range of 1450 nm – 1550 nm. The photonic band structure was calculated with the plane wave expansion method[8].

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