Field and intensity correlations in amplifying random media

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We study local and nonlocal correlations of light transmitted through active random media. The conventional approach results in divergence of ensemble-averaged correlation functions due to the existence of lasing realizations. We introduce a conditional average for correlation functions by omitting the divergent realizations. Our numerical simulation reveals that amplification does not affect local spatial correlation. The nonlocal intensity correlations are strongly magnified due to selective enhancement of the contributions from long propagation paths. We also show that by increasing gain, the average mode linewidth can be made comparable to the average mode spacing. This implies that light transport through a diffusive random system with gain may exhibit some similarities with that through a localized passive system.

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Without a counterpart in the electronic system, the coherent amplification of light adds a new dimension to the fundamental study of mesoscopic wave transport. An inherent quantum/wave signature of mesoscopic transport is nonlocal intensity correlation,[1–3] which reflects the closeness to Anderson localization transition.[4] Light transport in an amplifying random medium experiences enhanced contribution from long paths,[5] which should have a profound effect on the nonlocal intensity correlation.

Due to formal similarity, it is tempting to treat a random system with gain as if it had "negative absorption,"[6] and directly adopt the results obtained for an absorbing system. Such a simplistic approach to correlation functions (CFs) is fundamentally flawed. Theoretically, the spatial and spectral CFs are obtained by average over an infinite number of random realizations. Among them, there exist rare configurations containing more localized modes that could lase in the presence of gain. Light intensity in the lasing configurations diverges, and so do the ensemble-averaged CFs. Experimentally, the divergence of laser intensity is prevented by gain saturation. Nevertheless, the lasing configurations have much higher intensity than the nonlasing ones, thus they dominate the CFs. The width of spectral CFs is simply equal to the lasing mode linewidth, while the spatial CFs only reflect the spatial extent of the lasing modes. This is in contrast to the "negative absorption" model, which does not contain the divergent contribution of the rare events.[7] The analytical theories are based on perturbation approaches which implicitly drop long-path contributions described by high-order terms. In order to obtain the CFs that reflect light transport in amplifying random media, we introduce the conditional average over all nonlasing configurations $\langle\cdots\rangle \rightarrow \langle\cdots\rangle_c$. Such replacement, together with the fact that the fraction of lasing configurations *varies with the amount of gain*, makes any analytical derivation challenging. Numerical simulations turned out to be a fruitful alternative.

In this paper, based on numerical simulations, we present a phenomenological analysis of the local and nonlocal correlations of light transmitted through active random media. The systems under consideration are in the diffusive regime, but not too far from the localization threshold. We show that "negative absorption" formulas give a good fit to the conditional CFs only at low gain; with a decrease of the dimensionless conductance g this range of applicability is further reduced. For high gain, even after discarding the contributions of lasing configurations, the long-range correlations are significantly stronger than the prediction of "negative absorption" theory, especially for the systems closer to the localization threshold. At first glance, the removal of all lasing configurations, which are dominated by more localized modes and have stronger nonlocal correlations, should have weakened the long-range correlations averaged over the rest nonlasing (less localized) configurations. Moreover, the average mode linewidth $\delta\nu$, found from the width of conditional spectral CF, does not exhibit any widening compared to the "negative absorption" expression. This is unexpected because exclusion of the narrowest modes (which have lased) should have overestimated the "average" mode linewidth. Therefore, the enhancement of long-range correlations and narrowing of the conditional spectral correlation width caused by coherent amplification exceed the expectations from "negative absorption" model. It reveals the absence of duality between gain and absorption. We also calculate the effective Thouless number $\delta \equiv \delta\nu / d\nu$, where $d\nu$ is the average mode spacing. In the absence of gain, the onset of localization is marked by $\delta = 1$. We show that for diffusive systems, as gain increases, δ decreases monotonically to below 1 before reaching the diffusive lasing threshold predicted by the "negative absorption" theory that ignores fluctuations. This is an intriguing result, which seems to imply that transport in a diffusive system with gain could exhibit some similarities to that through a localized passive system.

In our numerical simulation, we consider 2D random medium in a waveguide geometry, shown in the inset of Fig. 1(a). It consists of a metallic waveguide filled with circular dielectric scatterers of refractive index $n = 2$. Our numerical method for calculation of CFs in passive systems has been described elsewhere.[8] Physically, our system is quasi-one-dimensional, and the transition from

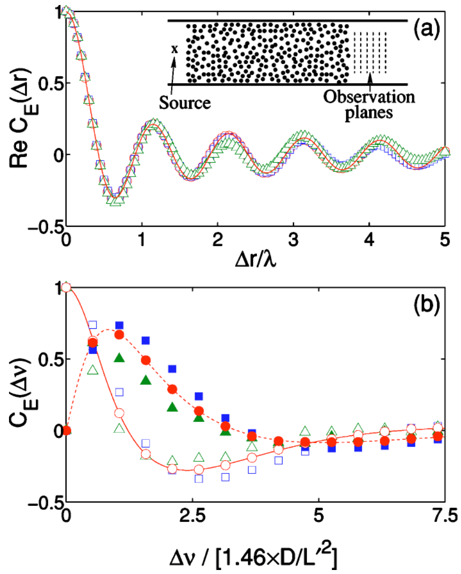

FIG. 1. (a) The real part of $C_E(\Delta r)$ and (b) the real (open symbols, solid line) and imaginary (solid symbols, dashed line) parts of $C_E(\Delta\nu)$. Circles, squares, and triangles represent numerical data for passive ($\tau_{abs/amp}=\infty$), absorbing ($\tau_{abs}=-5\tau_{amp}^{cr}$), and amplifying ($\tau_{amp}=5\tau_{amp}^{cr}$) systems, respectively. τ_{amp}^{cr} is the critical amplification time constant at the diffusive lasing threshold. $L=0.4$ m, $W=0.2$ m, $l=1.8$ cm. Curves represent theoretical fit with $z_b/l=0.8$. The inset is a sketch of the numerical experiment.

diffusion to localization can be realized by increasing the length L of the random medium. To demonstrate the independence of our results on microscopic structure of the system, we varied both the filling fraction of scatterers and the length of the random medium to obtain samples with dimensionless conductance, g, from 2.2 to 9. The effect of absorption or gain (inside the scatterers) is treated by a classical Lorenzian model[9] with positive or negative conductivity. The advantage of our numerical model is the ability to introduce spatially uniform gain as well as to separate coherent amplification of an input signal from spontaneous emission of the active medium. In the numerical experiment, a short pulse was launched via a point source at the input end of the waveguide. The Fourier transform of the electromagnetic field at the output end gave the CFs for field[10] $C_E(\Delta r,\Delta\nu)=\langle E(\mathbf{r}+\Delta\mathbf{r},\nu+\Delta\nu)E^*(\mathbf{r},\nu)\rangle/[\langle I(\mathbf{r}+\Delta\mathbf{r},\nu+\Delta\nu)\rangle^{1/2}\langle I(\mathbf{r},\nu)\rangle^{1/2}]$ and intensity[3] $C(\Delta r,\Delta\nu)=\langle I(\mathbf{r}+\Delta\mathbf{r},\nu+\Delta\nu)I(\mathbf{r},\nu)\rangle/[\langle I(\mathbf{r}+\Delta\mathbf{r},\nu+\Delta\nu)\rangle\langle I(\mathbf{r},\nu)\rangle]-1$ (see Ref. 8 for details on numerical statistical averaging). In the presence of gain, long after the short excitation pulse, the electromagnetic field decays with time in the nonlasing realizations, while it keeps increasing in the lasing ones. We excluded the lasing realizations from the ensemble average for CFs.

Based on the pairing of incoming and outgoing channels, three[11] contributions to intensity CF have been identified:[3] a local (short-range) $C_1=|C_E|^2$,[12] and two nonlocal C_2 (long-range) and C_3 (infinite-range) ones. For diffusive transport $g\gg 1$, in a waveguide geometry $C_1\sim 1$, $C_2\sim 1/g$, and $C_3\sim 1/g^2$, making the values of C_2 and C_3 small.[3] The nonlocal terms are brought about by long propagation paths which are most sensitive to the effect of amplification. Despite their small values in the passive systems, we observe a dramatic enhancement of the nonlocal correlations by coherent amplification.

The spatial field CF in 3D bulk was originally found by Shapiro in Ref. 10. Later Eliyahu *et al.*[13] calculated it at the output surface of 3D random medium. Similarly, we derived the corresponding expression in the 2D case:[8]

$$C_E(\Delta r)=\frac{\pi(z_b/l)J_0(k\Delta r)+2\sin(k\Delta r)/k\Delta r}{\pi z_b/l+2}, \qquad (1)$$

where $k=2\pi/\lambda$, l is the transport mean free path, J_0 is Bessel function of zeroth order, and the extrapolation length $z_b\sim l$ accounts for boundary effects. The imaginary part of $C_E(\Delta r)$ should vanish due to isotropy,[14] which is confirmed by our calculation where its value is less than 10^{-3}. The real parts of $C_E(\Delta r)$ are found unchanged in the presence of gain or absorption, as shown in Fig. 1(a) for a system of $g=2.2$. Equation (1) gives excellent fit for passive, absorbing, and amplifying systems with the same value of z_b/l. Physically, this invariance can be explained by the local nature of $C_E(\Delta r)$. The spatial field correlation contains information that comes from the length scale of order of the mean free path. l is always shorter than the ballistic gain length l_g: $l_g/l>(2n_{\text{eff}}^{(e)}/\pi^2)\cdot(L/l)^2\gg 1$, because the system is below the diffusive lasing threshold $L<\pi l_{amp}$. In the above expressions, the amplification length $l_{amp}=\sqrt{D\tau_{amp}}$, where D is the diffusion coefficient and $\tau_{amp}=l_g/c$, the effective index of refraction $n_{\text{eff}}^{(e)}=c/v_E$, where v_E is the energy transport velocity. Since amplification occurs on the scale much longer than l, it has negligible effect on short-range transport and local spatial correlations.

The spectral field CF $C_E(\Delta\nu)$ contains an important dynamical parameter of transport—the diffusion coefficient $D=v_E l/2$. The conditional spectral correlation width $\delta\nu$ is defined as the width at half maximum of $|C_E(\Delta\nu)|^2$ divided by a numerical factor 1.46.[8] In a passive system, $\delta\nu$ is equal to the average mode linewidth D/L'^2, where $L'=L+2z_b$. Since v_E can be determined separately through calculation of energy distribution between air and dielectric scatterers,[8] the transport mean free path was found by fitting of the real and imaginary parts of $C_E(\Delta\nu)$ [Fig. 1(b)]. The value of l allowed us to determine $g=(\pi/2)n_{\text{eff}}^{(e)}Nl/L'$, where $N=2W/\lambda\simeq 20-40$ is the number of propagating modes in the waveguide, and W is its width. In the presence of absorption,[15] the numerically calculated $C_E(\Delta\nu)$ fits well the expression derived in Ref. 16. For the case of amplification, we obtained the "negative absorption" formula by making the substitution $-\tau_{abs}\to\tau_{amp}$:

$$C_E(\Delta\nu)=\frac{\sinh(q_0 a)}{\sinh(q_0 L')}\frac{\sin(L'/l_{amp})}{\sin(a/l_{amp})}, \qquad (2)$$

where $q_0=\gamma_+-i\gamma_-$, $\gamma_\pm^2=(\sqrt{1/l_{amp}^4+\beta^4}\mp 1/l_{amp}^2)/2$, $\beta=\sqrt{2\pi\Delta\nu/D}$, and $a\simeq l$ is randomization length.[3] By fitting $C_E(\Delta\nu)$ with Eq. (2), we obtained $\delta\nu$, which is plotted in Fig. 2 for systems of $g=4.4$ and 9.0. The narrowing of the conditional spectral correlation width by gain is due to partial compensation of light leakage through the system bound-

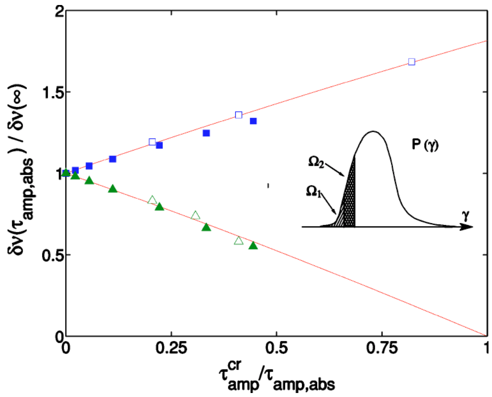

FIG. 2. The conditional spectral correlation width $\delta\nu$ as a function of amplification time τ_{amp} (triangles) or absorption time τ_{abs} (squares). $\delta\nu$ is normalized to the value without gain or absorption ($\tau_{amp/abs}=\infty$), $\tau_{amp/abs}$ to τ^{cr}_{amp}. Solid symbols correspond to $g=4.4$, open symbols to $g=9$. The solid curves are given by the diffusion theory. The inset schematically shows the distribution of the resonant mode linewidth (see text for discussion).

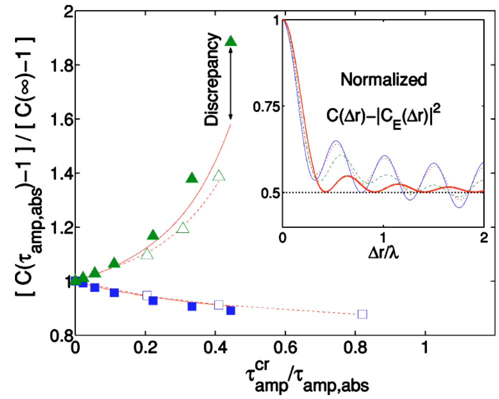

FIG. 3. $C-C_1$ at $\Delta r=0$ and $\Delta\nu=0$ in absorbing (squares) and amplifying (triangles) systems. Solid symbols correspond to $g=4.4$, open symbols to $g=9$. Solid and dashed curves are obtained from Eq. (3) without any fitting parameters. The inset compares the dependence of $C-C_1$ on Δr with $[F(\Delta r)+1]/2$ (thick line). Thin solid, dotted, and dashed lines represent passive, absorbing, and amplifying systems as in Fig. 1.

aries. Absorption, on the contrary, introduces an additional loss mechanism, which leads to an increase of $\delta\nu$. For both amplifying and absorbing media, the calculated $\delta\nu$ agrees well with the diffusion prediction. This agreement is surprising for the case of amplification. The "negative absorption" theory neglects the fluctuation of lasing threshold, and assumes the spectral width of all modes decreases with gain uniformly. However, the width γ of resonant modes has a distribution $P(\gamma)$, schematically plotted in the inset of Fig. 2. For a given amount of gain, the modes with small γ in the tail (Ω_1) of $P(\gamma)$ lase, and they are excluded from the ensemble average. Such selective elimination of the narrowest modes should have led to an overestimation of $\delta\nu$. The absence of deviation from Eq. (2) indicates amplification not only reduces the width of all nonlasing modes, but also enhances the weight of the modes with narrower-than-average width (in Ω_2) in the averaging.

In Fig. 2, the diminishing correlation width in the amplifying system signifies an approach to the lasing threshold for the mode with average linewidth. According to Eq. (2), $\delta\nu=0$ when $\sin(L'/l_{amp})$ turns to zero at $L'/l_{amp}=\pi$. This "average" lasing threshold agrees with the diffusive lasing threshold derived by Letokhov.[17] Our calculation, Fig. 2, shows that the (conditional) average mode linewidth $\delta\nu$ can become comparable to the average mode spacing $d\nu$ before the diffusive lasing threshold is reached. Namely, with increasing gain, $\delta\nu$ decreases to $d\nu$ before reaching zero. This means the effective Thouless number δ can be reduced toward unity by coherent amplification for a system that is diffusive in the absence of gain.

Figure 3 shows the nonlocal part of spatial intensity CF, $C(\Delta r,\tau^{cr}_{amp}/\tau_{amp})-|C_E(\Delta r,\tau^{cr}_{amp}/\tau_{amp})|^2$, in samples of $g=4.4$ and 9.0. According to Refs. 12 and 18, spatial variation and absorption contribution should factorize. Accounting for terms up to $1/g^2$ we obtained the "negative absorption" expression for nonlocal intensity correlation,[15,16,18–20]

$$C(\Delta r,s=L/l_{amp})-|C_E(\Delta r,s)|^2=(1+F(\Delta r)) \times\left[\frac{1}{4gs}\frac{2s(2-\cos 2s)-\sin 2s}{\sin^2 s}+\frac{4}{g^2}\frac{\sin^2 s}{s^2}\times\left(\frac{2s^2-s\cot s+1}{16\sin^2 s}-3\frac{s^2+s\cot s+1}{16\sin^4 s}+\frac{3s^2}{8\sin^6 s}\right)\right], \quad (3)$$

where $F(\Delta r)=|C_E(\Delta r)|^2$. The inset of Fig. 3 plots the profile of $C(\Delta r)-C_1(\Delta r)$, normalized to its value at $\Delta r=0$. For passive, absorbing, and amplifying systems, the dependence of $C-C_1$ on Δr is almost the same. In particular, the value of $C-C_1$ at $\Delta r\to\infty$ is exactly 1/2 of that at $\Delta r=0$, in agreement with $[1+F(\Delta r)]$ dependence. This tells us amplification (absorption) increases (decreases) the nonlocal correlations

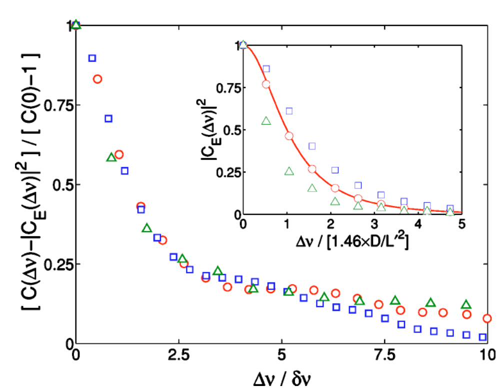

FIG. 4. The frequency dependence of the nonlocal contribution to $C(\Delta\nu)$ normalized to its value at $\Delta\nu=0$. The frequency difference, $\Delta\nu$, is expressed in units of the correlation linewidth, $\delta\nu$, determined from the corresponding $C_1(\Delta\nu)=|C_E(\Delta\nu)|^2$ shown in the inset. System parameters and symbol notations are the same as in Fig. 1.

at every Δr uniformly. Therefore, the enhancement (reduction) can be characterized by a single number, e.g., the value of $C-C_1$ at $\Delta r=0$ as shown in Fig. 3. In two absorbing samples of $g=4.4$ and 9, the decrease of nonlocal correlations is in good agreement with the diffusion prediction. For amplifying media, only when the fraction of omitted lasing realizations is small does Eq. (3) adequately describe the nonlocal correlations of the transmitted intensity. For high gain, we see strong deviations: even after removing the lasing realizations, nonlocal correlation still exceeds the "negative absorption" prediction [Eq. (3)]. The deviation becomes more pronounced as the dimensionless conductance decreases. The rapid increase of nonlocal correlation with gain is caused by enhanced contribution from long trajectories that cross upon themselves.

Finally, we calculated the spectral correlations of transmitted intensities. Figure 4 reveals the changes of $C_1(\Delta\nu)$ and $C(\Delta\nu)-C_1(\Delta\nu)$ due to gain and absorption. $C(\Delta\nu)-C_1(\Delta\nu)$ is normalized to its value at $\Delta\nu=0$. Similar to the effect of localization[8] C_1 and $C-C_1$ are narrowed by gain. In the case of localization, the narrowing occurs as the system length is increased, while in the case of gain, it occurs in the sample with fixed length. In the main part of Fig. 4, we choose to normalize the frequency separation $\Delta\nu$ by the corresponding linewidth of $C_1(\Delta\nu)$, $\delta\nu$. We can see that the correlation width of the nonlocal terms diminishes (within numerical accuracy) at the same rate as that of C_1. In contrast, in passive systems at the onset of localization, we found[8] slower decrease of the width of $C(\Delta\nu)-C_1(\Delta\nu)$ contribution. This shows that when a passive system approaches the localization transition, the more conductive channels dominate the nonlocal correlations, whereas in a diffusive system with gain, the modes with narrow widths are preferably amplified and lead to stronger decrease of the width of $C(\Delta\nu)-C_1(\Delta\nu)$.

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